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Robustness and Performance Parameterization of Smooth Second Order Sliding Mode Control

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Abstract: Novel robustness and performance parameters are established for Smooth Super Twisting Algorithm (SSTA). The stability of SSTA is well established for arbitrary gains using homogeneity approach. The design and tuning of the controller parameters is a major issue and no analytic design method is available so far. A novel Lyapunov function is proposed and by the virtue of stability analysis, the stability bounds for a certain class of uncertainties are determined. In addition, the issue of finite time convergence is also explored, resulting in determination of the settling time as a function of the controller parameters. The proposed settling time formulation suggests a methodical approach to SSTA design in contrast to the available rules of thumb. Unlike the literature available for Higher Order Sliding Mode (HOSM) controllers, the proposed design framework is validated against a challenging problem of the Underground Coal Gasification (UCG) process control. Like the other process control problems the chosen problem is nonlinear and contains significant uncertainties.

Keywords: Discontinuous system, higher order sliding mode (HOSM), smooth supertwisting algorithm (SSTA), underground coal gasification (UCG).

1. INTRODUCTION

One of the most stimulating aspect of the Sliding Mode Control (SMC) [1–4] is the discontinuous nature of the controller, whose primary function is to switch between two characteristically different system structures such that a new type of system motion, called sliding modes [5], exists in a manifold, known as the sliding manifold. This inspirational system phenomenon results in splendid system performance, which includes parameter invariance and remarkable robustness against disturbances and model uncertainties. However, enforcing sliding modes require the discontinuous controller to perform switching at an infinite frequency which is not possible for physical systems. This limitation causes the high frequency oscillations, known as chattering, about the sliding manifold. The chattering phenomenon is dangerous for actuators and in some cases it causes total system failure. Another limitation of the conventional SMC is that it is only applicable to the relative degree one systems (control appears explicitly in the first derivative of the sliding manifold [5]).

The control system community continued the research, with the aim to keep intact all the benefits (robustness and parameter invariance) of the SMC while avoiding or minimizing the chattering. A number of techniques have been proposed for chattering avoidance, such as, low pass filtering of the control signal, approximation of *signum* function by a *saturation* function and the HOSM control [6–8]. The HOSM controllers such as Super Twisting and Real Twisting Algorithms (STA and RTA respectively) gained popularity in the community of control theoreticians and practitioners, due to their generalized structures, applicability to relative degree one and two systems respectively and more importantly, reduced chattering.

In a multi-loop system these controllers do not offer much performance improvement in the situation where an inner loop requires a continuous signal from the outer loop. To overcome this inherent flaw of STA and RTA, a smooth second order SMC framework was proposed in [9]. In [10], a smooth second order SMC based on STA was proposed and its stability was proved using the homogeneity approach. The smooth control algorithms proved to be effective in multi loop control schemes, as they produce a continuous (smooth or almost chattering free) control action.

One of the most important problems in SMC is the stability analysis and proving the finite time convergence of the algorithm. For the stability and finite time convergence of STA, SSTA, RTA and Smooth RTA (SRTA) the geometric [7], or homogeneity [11], approaches are used. These approaches did well in stability analysis of these algorithms but did not provide the robustness bounds or

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estimate of the convergence time. In Moreno *et al.* [12], a Lyapunov function was used for the first time for the stability analysis and finite time convergence of STA. This approach allows to study this algorithm more deeply as it parametrizes the algorithm's stability and convergence time. In [13], the theory developed in [12], was used to set a linear framework for the stability analysis of STA. In Moreno *et al.* [14], a method for constructing strict Lyapunov function was proposed which was computationally very effective and efficient. Furthermore, the stability and robustness analysis of STA was carried out using this technique.

The closed loop system with SSTA proposed in [10] has the structure shown in (1) below:

$$\begin{aligned} \dot{x}_1 &= -k_1|x_1|^{\frac{\rho-1}{\rho}} sign(x_1) + x_2 + \zeta_1(t,x), \\ \dot{x}_2 &= -k_2|x_1|^{\frac{\rho-2}{\rho}} sign(x_1) + \zeta_2(t,x), \end{aligned} \tag{1}$$

where $x \in \Re^2$ is the state vector, k_i, $i = 1,2$ are the controller gains, $\rho \geq 2$ is the smoothing parameter and $\zeta_i(t,x)$ are bounded matched disturbances.

Since SSTA is very sensitive to the so-called drift terms (see [10] and [15]), so the authors used a second order disturbance observer to cancel the effects of these terms. The homogeneity approach used in [10] for the analysis of system (1) does prove the stability subjected to the drift term. However, this approach does not provide any analytical solution for the gains and convergence time of the closed loop system. In addition, when system (1) is subjected to matched disturbance then the homogeneity in closed loop becomes a question.

In this paper, we present a novel robust stability analysis of SSTA in closed loop. The main contributions are:

- Development of a strict Lyapunov function for the robust stability of SSTA.
- Analytical expressions for the gains of SSTA are developed.
- Analytical expressions are proposed for the convergence time determination of SSTA, which can be used for performance improvement of the closed loop system.
- The effects of the controller parameters k_1, k_2 and ρ, on the closed loop performance are discussed.
- SSTA, with the proposed analytical expressions for its gains, is applied to the challenging process control of UCG.

The paper is organized as follows: Section 2 covers the development of a strict Lyapunov function for the unperturbed system (1) and presents the proof of stability and finite time convergence. In Section 3, analysis of the closed loop system (1) subjected to bounded matched disturbance is carried out and a method for choosing the controller gains is proposed. In Section 4, SSTA is applied to the process control of UCG, for maintaining the calorific value of the product gases at a desired value and Section 5 concludes this work.

2. SSTA WITHOUT PERTURBATION

Consider system (1) to be nominal ($\zeta_1(t,x) = \zeta_2(t,x) = 0$) as given in (2)

$$\begin{aligned} \dot{x}_1 &= -k_1|x_1|^{\frac{\rho-1}{\rho}} sign(x_1) + x_2, \\ \dot{x}_2 &= -k_2|x_1|^{\frac{\rho-2}{\rho}} sign(x_1). \end{aligned} \tag{2}$$

Inspired by the technique proposed in [12] and [14], take a quadratic, positive definite and radially unbounded Strict Lyapunov function:

$$V(x) = \xi^T P \xi, \tag{3}$$

where $\xi^T = [|x_1|^y sign(x_1) \quad x_2]$, $y = \frac{\rho-1}{\rho}$ and $P \in \Re^{2\times 2}$ is symmetric and positive definite matrix, which is a solution of the arithmetic Lyapunov equation (ALE) (4).

$$A^T P + PA = -Q. \tag{4}$$

Here $Q \in \Re^{2\times 2}$ is another symmetric and positive definite matrix.

The time derivative of the vector ξ along the trajectories of (2), is given by:

$$\begin{aligned} \dot{\xi} &= \begin{bmatrix} \frac{\partial}{\partial t}|x_1|^y sign(x_1) & \frac{\partial}{\partial t}x_2 \end{bmatrix}^T \\ &= \begin{bmatrix} y|x_1|^{y-1}\dot{x}_1 & \dot{x}_2 \end{bmatrix}^T \\ &= \begin{bmatrix} y|x_1|^{y-1}(-k_1|x_1|^y sign(x_1) + x_2) \\ -k_2|x_1|^M sign(x_1) \end{bmatrix} \end{aligned}$$

where $\dot{x}_1$ and $\dot{x}_2$ are given in (2) and $M = [(\rho-2)/\rho] = y - (1/\rho)$ simplifies the equation to give

$$\begin{aligned} \dot{\xi} &= |x_1|^{-\frac{1}{\rho}} A \xi, \\ \dot{\xi}^T &= |x_1|^{-\frac{1}{\rho}} A^T \xi^T, \end{aligned} \tag{5}$$

where

$$A = \begin{bmatrix} -yk_1 & y \\ -k_2 & 0 \end{bmatrix}.$$

So the time derivative of (3) along the trajectories of (2) and using (4) and (5) is given by:

$$\begin{aligned} \dot{V} &= \dot{\xi}^T P \xi + \xi^T P \dot{\xi} \\ &= |x_1|^{-\frac{1}{\rho}} \left[\xi^T A^T P \xi + \xi^T P A \xi\right] \\ &= |x_1|^{-\frac{1}{\rho}} \xi^T \left[A^T P + PA\right] \xi \\ &= -|x_1|^{-\frac{1}{\rho}} \xi^T Q \xi. \end{aligned} \tag{6}$$

The end result in (6) will be negative semi definite if and only if the matrix Q is positive definite. Following the definition of the ALE (4), Q will be symmetric and positive definite if and only if matrix A is Hurwitz because the matrix P is already assumed to be symmetric and positive definite. Furthermore, A will be Hurwitz if and only if the controller gains k_1 and k_2 are non-negative. So the stability of (2) is completely determined by the stability of the matrix A similar to the linear time invariant systems.

This fact is illustrated in Theorem 1, which associates the stability of the origin to the stability of matrix A.

Theorem 1: Consider (2), with gains $k_1 > 0$, $k_2 > 0$, then the following statements are claimed:

- Origin is unique finite-time stable equilibrium point.
- System (2) enforces a second order sliding mode control.
- For any arbitrary symmetric and positive definite matrices Q and P, any trajectory starting at initial condition x_0 reaches the origin in time less than T_s, which is given by:

$$T_s = \frac{\rho \lambda_{max}[P]}{\lambda_{min}^{\frac{1}{\rho}}[P]\lambda_{min}[Q]} V^{1/\rho}(x_0),$$

where $\lambda_{max}[P]$, $\lambda_{min}[P]$ and $\lambda_{min}[Q]$ are the eigenvalues of matrices P and Q respectively.

Proof: As system (2) has a discontinuous right hand side, so this system is understood in Filippov sense [16], and since it is a differential inclusion $\dot{x} \in f(x)$ and $0 \in f(0)$ [17], so origin is an equilibrium point for (2). Then consider (6), $\dot{V}$ is negative definite if and only if Q is positive definite. It follows from the definition of ALE (4) that if A is Hurwitz and Q is symmetric and positive definite only then there exist a unique symmetric and positive definite matrix P [19]. With this P, (3) becomes positive definite and its derivative becomes negative definite, which proves the stability of the origin.

Consider again (6), and applying invariant set theorem [18], $\dot{V} = 0$ on the set

$$R = \{(x_1, x_2) \in \Re^2 \quad | \quad x_1 = 0\},$$

and the largest invariant set ℓ in R is

$$\ell = \{(x_1, x_2) \in \Re^2 \quad | \quad x_1 = x_2 = 0\},$$

which proves that (2) exhibits second order sliding modes.

As system (2) is stable and origin is the only equilibrium point, which guarantee that the states $x(T_s) = 0$ and hence $V(x(T_s)) = 0$, where T_s is the settling time. For the initial conditions $x(0) = x_0$ and $V(x_0) = V_0$, the following inequalities are true [18].

$$\begin{aligned} \lambda_{min}[P]\,\|\xi\|_2^2 \leq & V \leq \lambda_{max}[P]\,\|\xi\|_2^2, \\ \|\xi\|_2^2 \geq & \frac{V}{\lambda_{max}[P]}, \\ \|\xi\|_2^2 \leq & \frac{V}{\lambda_{min}[P]}, \end{aligned} \tag{7}$$

and

$$\begin{aligned} &|x_1| \leq \|\xi\|_2^2, \\ &|x_1|^{\frac{1}{\rho}} \leq \left[\frac{V}{\lambda_{min}[P]}\right]^{\frac{1}{\rho}}. \end{aligned} \tag{8}$$

Using (7) and (8) in (6), we get

$$\dot{V} \leq -\delta V^{y}, \tag{9}$$

where,

$$\delta = \frac{\lambda_{min}^{\frac{1}{\rho}}[P]\lambda_{min}[Q]}{\lambda_{max}[P]}.$$

Now using Bihari's inequality [20] and the idea of separable differential equations [21] on (9)

$$V(x(t)) \leq \left[-\frac{\delta}{\rho}t + V_0^{\frac{1}{\rho}}\right]^{\rho},$$

which gives

$$T_s \leq \frac{\rho}{\delta} V_0^{\frac{1}{\rho}}.$$

□

Remark 1: The proof of Theorem 1 clearly shows the dependence of settling time on the smoothening parameter ρ and the controller gains.

Remark 2: The use of quadratic strict Lyapunov function has allowed to study the stability and performance of a discontinuous system, similar to a continuous, linear and time invariant system.

3. SSTA WITH PERTURBATIONS

In order to explore the stability of the perturbed closed loop system (1), assume that:

Assumption 1: $\zeta_1(t,x) = 0$ and $\zeta_2(t,x)$ is a vanishing perturbation i.e.,

$$\begin{aligned} &\zeta_2(t,x) = 0 \quad \forall\, x = 0, \quad t \in [0,\infty), \\ &|\zeta_2(t,x)| \leq L \quad \forall\, x \neq 0, \quad t \in [0,\infty). \end{aligned} \tag{10}$$

Based on this assumption, a selection rule is proposed for the gains k_1 and k_2, such that the effect of these perturbations will be nullified.

Theorem 2: Consider the perturbation terms in (1) satisfy (10), then there exist, constant, symmetric and positive definite matrices P and Q_p such that (3) is positive definite and

$$\dot{V} = -|x_1|^{-\frac{1}{\rho}} \xi^T Q_p \xi$$

is globally semi negative definite. Moreover, with a proper selection of gains k_1 and k_2, the origin is a global finite time stable equilibrium point with settling time less than T_{ps}.

$$T_{ps} = \frac{\rho}{\delta_p} V_0^{\frac{1}{\rho}},$$

where

$$\delta_p = \frac{\lambda_{min}^{\frac{1}{\rho}}[P]\,\lambda_{min}[Q_p]}{\lambda max[P]}.$$

Proof: Taking the time derivative of (3), along the trajectories of system (1) with the perturbation terms satisfying (10).

$$\dot{V} = |x_1|^{-\frac{1}{\rho}} \left[\xi^T A^T P\xi + \xi^T PA\xi\right] + \dots \\ + |x_1|^{-\frac{1}{\rho}} \left[\zeta^T P\xi + \xi^T P\zeta\right], \tag{11}$$

where

$$\zeta = \begin{bmatrix} 0 & |x_1|^{\frac{1}{\rho}} \zeta_2(t,x) \end{bmatrix}^T, \tag{12}$$

and the arbitrary P and Q_p are

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad Q_p = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix},$$

where $P_{12} = P_{21}$ and $Q_{12} = Q_{21}$. Then (11) becomes

$$\dot{V}(x) = |x_1|^{-\frac{1}{\rho}} \xi^T \left[(A+M)^T P + P(A+M)\right] \xi,$$

where

$$M(t,x) = \begin{bmatrix} 0 & 0 \\ n(t,x) & 0 \end{bmatrix}, \tag{13}$$

and $n(t,x) = \zeta_2(t,x)\,|x_1|^{-\frac{\rho-2}{\rho}} sign(x_1)$. From (10), $n(t,x)$ is bounded. With these, $\dot{V}(x)$ will be negative definite if and only if:

- P is symmetric and positive definite.
- $A+M$ is Hurwitz.
- The ALE $(A+M)^T P + P(A+M) = -Q_p$ is satisfied for some symmetric and positive definite matrix Q_p.

These conditions will be satisfied if the following inequalities are true.

$$k_1 > 0\,,\; k_2 > L\,,\; P_{22} > yP_{12}^2 > 0, \\ P_{12} < 0\,,\; P_{11} > 0, \\ 0 < -4yk_1P_{12} - 4y(k_2 - n)P_{12}^2 \dots \\ \dots - (yk_1P_{12} - 1)^2 - (k_2 - n)^2 P_{12}^2 \dots \\ \dots - 2(yk_1P_{12} - 1)(k_2 - n)P_{22}. \tag{14}$$

Now using the fact that $(k_2 - L) \le (k_2 - n(t,x)) \le (k_2 + L)$ and $k_2 > L$, the last inequality in (14) can be re-written as in (15).

$$0 < -4yk_1P_{12} - 4y(k_2 - L)P_{12}^2 \dots \\ \dots - (yk_1P_{12} - 1)^2 - (k_2 - L)^2 P_{12}^2 \dots \\ \dots - 2(yk_1P_{12} - 1)(k_2 - L)P_{22}. \tag{15}$$

The settling time for the perturbed system is then calculated the same way as in the proof of Theorem 1. □

From the proof of Theorem 2 above, we derive matrices P, Q_p and gains k_1 and k_2 satisfying (14) and (15), according to the following algorithm.

1) Choose the positive constants β and γ such that $0 < \beta < 1$, $\gamma > 1$ and $\beta\gamma > 2$.
2) Choose $\psi = y(k_2 + L)P_{22}$ and $\chi = -yk_1P_{12}$, such that the following inequality is satisfied.

$$(\chi + 1)^2 + \psi^2 < 4\chi - 4y\frac{\Psi}{\gamma} + \dots \\ + 2y(\chi + 1)\Psi\beta. \tag{16}$$

The inequality (16) represents the interior of an ellipse (sub-level sets) in the plane (χ, ψ), with center point (χ_c, ψ_c) and is characterized by β, γ and ρ, as shown in Fig. 1.

$$\chi_c = 1, \psi_c = y\left(\frac{\beta\gamma - 2}{\gamma}\right). \tag{17}$$

So if $\beta\gamma > 2$ is satisfied then the center of the ellipse is in the first quadrant.

3) For parameters χ, ψ, β and γ, we derive the matrices P and Q_p as:

$$P_{11} = \frac{1}{y}\,,\; P_{12} = P_{21} = -\sqrt{\frac{(1-\beta)\psi}{2\gamma L}},$$

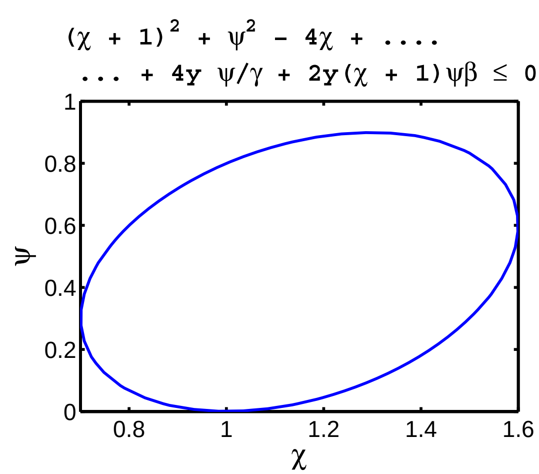

Fig. 1. Ellipse describing the boundary of set (16) with $\rho = 3$, $\beta = 0.5$ and $\gamma = 5$.

$$P_{22} = \gamma P_{12}^2\,,\; Q_{11} = 2\left(k_1 + (k_2 - L)\,P_{12}\right),$$

$$Q_{12} = Q_{21} = y\,(k_1 P_{12} - 1) + (k_2 - L)\,P_{22},$$

$$Q_{22} = -2yP_{12}. \tag{18}$$

From the above algorithm one can derive the gains k_1 and k_2 as:

$$k_1 = \frac{1}{y}\chi\sqrt{\frac{2\gamma L}{(1-\beta)\,\psi}},$$
$$k_2 = L\left[\frac{\rho\,(1+\beta)-\beta}{\rho\,(1-\beta)+\beta}\right]. \tag{19}$$

In Theorem 2 it is shown that SSTA is robust against perturbations satisfying (10).

Remark 3: For the convergence of a closed loop system with an SMC in the loop, it is necessary that the controller gain must be greater than the upper bound (L) of the disturbances. So the question that how much greater must be the gains of SSTA is answered by (19).

In the next theorem it is shown that SSTA will also endure the disturbance when $\zeta_1(t,x) \neq 0$, and is bounded by

$$|\zeta_1(t,x)| \leq \alpha_1 + \alpha_2 \,\|\xi\|_2\,, \tag{20}$$

for some non-negative constants α_1 and α_2. From (20) it is clear that if $\alpha_1 = 0$, then $\zeta_1(t,x)$ vanishes at the origin and the trajectories converge at the origin in finite time. If $\alpha_1 \neq 0$ then $\zeta_1(t,x)$ does not vanish at the origin, and consequently the trajectories of the perturbed closed loop system (1) will be globally ultimately bounded [18–20].

Theorem 3: Consider system (1) with the perturbation terms $\zeta_1(t,x)$, $\zeta_2(t,x)$ and gains k_1, k_2 satisfying (20), (10) and (19) respectively. Then the trajectories of the perturbed system (1) are globally ultimately bounded by:

$$b = \sqrt{\frac{\lambda_{max}[P]}{\lambda_{min}[P]}\frac{2\alpha_1\eta}{(1-\kappa)[\lambda_{min}[Q_p]-2\alpha_2\eta]}},$$

for $\alpha_2 \leq \frac{\lambda_{min}[Q_p]}{2\eta}$, $\alpha_1 > 0$, $\eta\Delta\sqrt{1+(yP_{12})^2}$, with P, Q_p given in (18) and $0<\kappa<1$. Moreover, if $\alpha_1 = 0$, then origin $x=0$ is a global finite time stable equilibrium point and all the trajectories converge to the origin in time less than $\overline{T}_{1s}$

$$\overline{T}_{1s} = \frac{\rho\lambda_{max}^{y}[P]}{\lambda_{min}[Q_p]-2\alpha_2\eta}V_0^{\frac{1}{\rho}},$$

where $V_0 = V(x_0)$ and x_0 is the initial condition. When $\alpha_1 \neq 0$ then instead of the origin, the trajectories will converge to the manifold Ω,

$$\Omega = \left\{x\in\Re^2 \quad | \quad V(x(t)) \leq \lambda_{max}[P]\mu^2\right\},$$

in time less than $\overline{T}_{2s}$

$$\overline{T}_{2s} = \rho\frac{\lambda_{max}^{y}[P]}{\kappa\,(\lambda_{min}[Q_p]-2\alpha_2\eta)}\left(V_0^{\frac{1}{\rho}} - \lambda_{max}^{\frac{1}{\rho}}[P]\mu^{\frac{2}{\rho}}\right),$$

where

$$\mu\Delta\frac{2\alpha_1\eta}{(1-\kappa)\,(\lambda_{min}[Q_p]-2\alpha_2\eta)}.$$

Proof: The time derivative of (3) along the trajectories of system (1) with the disturbance terms satisfying (10) and (20) is

$$\dot{V} = |x_1|^{\frac{1}{\rho}}\,[\xi^T A^T P\xi\xi^T PA\xi + B_1^T P\xi + \ldots$$
$$+\xi^T PB_1] + |x_1|^{\frac{1}{\rho}}\left[B_2^T P\xi + \xi^T PB_2\right], \tag{21}$$

where B_1 and B_2 are arbitrary given names

$$B_1 = \begin{bmatrix}0 & |x_1|^{\frac{1}{\rho}}\,\zeta_2\end{bmatrix}^T, B_2 = [y\zeta_1 \quad 0]^T.$$

Applying the results obtained in the proof of Theorem 2 to (21), we get:

$$\dot{V} = -|x_1|^{-\frac{1}{\rho}}\left[\xi^T Q_p\xi - B_2^T P\xi - \xi^T PB_2\right],$$
$$= -|x_1|^{-\frac{1}{\rho}}\left[\xi^T Q_p\xi - 2\zeta_1\xi^T\begin{bmatrix}1\\ yP_{12}\end{bmatrix}\right]. \tag{22}$$

Using the inequalities

$$\zeta_1\xi^T\begin{bmatrix}1\\ yP_{12}\end{bmatrix} \leq \zeta_1\,\|\xi\|_2\,\eta \leq (\alpha_1+\alpha_2\,\|\xi\|_2)\,\eta\,\|\xi\|_2\,,$$

and

$$\lambda_{min}[Q_p]\,\|\xi\|_2^2 \leq \xi^T Q_p\xi \leq \lambda_{max}[Q_p]\,\|\xi\|_2^2,$$

equation (22) becomes,

$$\dot{V} \leq -|x_1|^{-\frac{1}{\rho}}\left[g\,\|\xi\|_2^2 - 2\alpha_1\eta\,\|\xi\|_2\right], \tag{23}$$

where

$$g = (\lambda_{min}[Q_p]-2\alpha_2\eta).$$

Case 1: $\alpha_1 = 0$
In this case $\dot{V}(x)$ (23) will be negative definite if and only if $\lambda_{min}[Q_p] \geq 2\alpha_2\eta$.

$$\dot{V}(x) \leq -ZV^{y}(x),$$
$$Z = \frac{\lambda_{min}[Q_p]-2\alpha_2\eta}{\lambda_{max}^{y}[P]}. \tag{24}$$

Following the procedure in the proof of Theorem 1 for calculating the settling time we get $\overline{T}_{1s}$:

$$\overline{T}_{1s} = \rho\frac{\lambda_{max}^{y}[P]}{\lambda_{min}[Q_p]-2\alpha_2\eta}V_0^{\frac{1}{\rho}}.$$

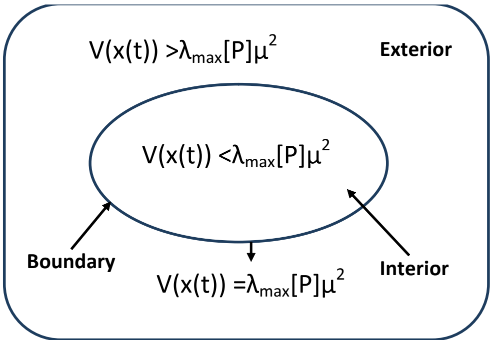

Fig. 2. Description of the Manifold (Ω).

Case 2: $\alpha_1 \neq 0$
In this case for $\dot{V}(x)$ to be negative definite we need

$$2\alpha_2\eta \leq \lambda_{min}[Q_p],$$
$$2\alpha_1\eta \leq (\lambda_{min}[Q_p] - 2\alpha_2\eta)(1-\kappa)\|\xi\|_2,$$

from these conditions we derive

$$\|\xi\|_2 \geq \mu. \tag{25}$$

Now using the identity in (7) and (25)

$$V(x(t)) \geq \lambda_{max}[P]\mu^2. \tag{26}$$

The inequality (26) justifies the manifold Ω, described in Fig. 2.

Evaluation of (23) at the boundary of the manifold Ω, gives:

$$\dot{V}(x) \leq (\lambda_{min}[Q_p] - 2\alpha_2\eta)\,\kappa\frac{V^y(x)}{\lambda_{max}^y[P]}. \tag{27}$$

In solving (27) for settling time $\overline{T}_{2s}$, keep in consideration the fact that now $V(x(\overline{T}_{2s})) \neq 0$ (because origin is not the equilibrium point) rather $V(x(\overline{T}_{2s})) \to \Omega$. In short $\overline{T}_{2s}$ is the time in which the trajectories of system (1) reach the manifold Ω. Under these conditions the solution for $\overline{T}_{2s}$ is performed using procedure in the proof of Theorem 1. □

Remark 4: Uniform ultimate boundedness is always the definition of stability when the system is under the effect of some non-vanishing perturbations. The final bound in this case will be a function of those terms which define the perturbation.

Remark 5: The proof of Theorem 3 shows the dependence of the settling on the bounds of perturbations and importantly on the smoothing parameter ρ i.e., settling times $\overline{T}_{1s}$ and $\overline{T}_{2s}$ are directly proportional to ρ.

In the next section, the SSTA with the proposed analytic expressions for gains k_1 and k_2 (19) are applied to the process control problem of UCG. The control purpose is to maintain a maximum calorific value of the product gases at the outlet well of the UCG reactor.

4. CONTROL OF THE PROCESS OF UCG

The process of UCG, aims at utilizing coal, deep inside the earth, for energy production. This is accomplished by drilling two wells called the injection well and the production well, from the surface of the earth to a coal seam. The process starts by igniting the coal. After ignition the pyrolysis reaction takes place to produce *char*. The oxidants (air/oxygen) and steam (H_2O) are injected through the injection well, which reacts chemically with *char* (gasification reaction) producing synthesis gas (a mixture of carbon mono oxide (CO) and hydrogen (H_2)). The synthesis gas can be used as a fuel in combined cycle turbines (CCT) for electricity generation or as a chemical feedstock [22, 23].

The mathematical model of a UCG reactor represents two phases: solid and gas. Solid phase consists of two species: coal and char, where as the gas phase is comprised of eight gases: CO, carbon dioxide (CO_2), Steam (H_2O), H_2, methane (CH_4), nitrogen (N_2), O_2 and Tar (higher hydrocarbons). In [24] the time domain model of UCG is developed based on the work of [25] and [26]. A generalized state space representation of UCG model is given in (28).

$$\begin{aligned}
\dot{x}_1 &= M_{coal}\sum_{j=1}^{3} a_{coal,j}r_j,\\
\dot{x}_2 &= M_{char}\sum_{j=1}^{3} a_{char,j}r_j,\\
\dot{x}_3 &= \frac{1}{C_s}\left[ht(T-x_3)-H_s\right],\\
\dot{x}_i &= \sum_{j=1}^{3} a_{gas,j}r_j - \beta x_i, \quad \text{where } i = 4,5,6,7,8,\\
\dot{x}_9 &= \sum_{j=1}^{3} a_{H_2O,j}r_j - \beta x_9 + \frac{a}{L}u + \frac{\zeta}{L},\\
\dot{x}_{10} &= \sum_{j=1}^{3} a_{O_2,j}r_j - \beta x_{10} + \frac{b}{L}u,\\
\dot{x}_{11} &= -\beta x_{11} + \frac{c}{L}u,\\
h &= mf_{CO}H_a + mf_{H_2}H_b + mf_{CH_4}H_c,
\end{aligned} \tag{28}$$

where the input $u \in \Re$ is the flow rate of inlet gases (a mixture of O_2, H_2O and N_2). The output (h) of the reactor, is the calorific value of the product gas. The water influx from the surrounding aquifers is an input disturbance to the process and is represented by $\zeta(t,x) \in \Re$. The profile for the water influx is shown in Fig. 3 having an upper bound: $l = 3\times10^{-5}\ moles/cm^2Sec$. The description of states, and other parameters of (28) is given in Table 1.

4.1. Control problem

The process of UCG is very sensitive to the moles of oxidants injected at the inlet well. In Fig. 4 the response of

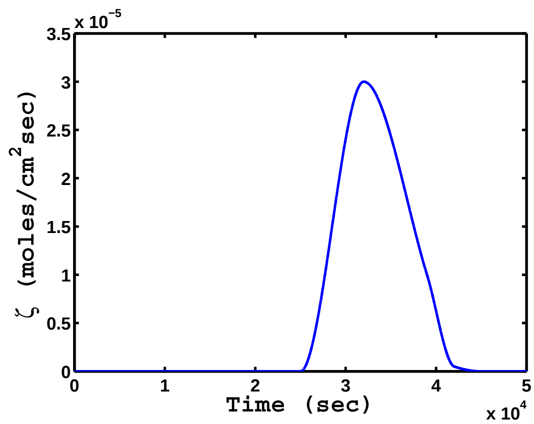

Fig. 3. The profile of water influx w.r.t time.

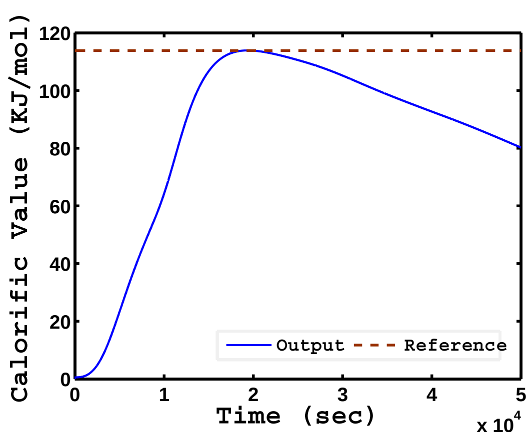

Fig. 4. Open loop response of the system w.r.t time.

the system for a constant input (flow rate of oxidants at the inlet) ($u = 2 \times 10^{-4}$moles/cm^{2}sec) is shown. The figure shows that the calorific value of product gases reaches a maximum value of 113.831 KJ/mol in 20,000 secs, and then starts decreasing. This decrease is due to the surplus amount of the moles of the inlet gas mixture. Therefore, the control problem is to control the flow rate in such a way that the maximum calorific value obtained at 20,000 sec, is maintained.

In order to achieve this objective, a bound on the control input, set by hardware (compressors) limitations, shown in (29), must not be violated.

$$0 < u \leq 3 \times 10^{-4}. \quad (29)$$

4.2. Simulation results

Based on the results obtained in Section 3, the following parameters of SSTA are chosen for the simulations: $\beta = 0.80$, $\rho = 3$ and $\gamma = 3$, while the parameters χ and ψ are selected as the center points of the ellipse (16), i.e. ($\chi = 1$, $\psi = 0.09$). These values give gains, $k_1 = 0.15$ and $k_2 = 10^{-4}$.

The simulations are carried out using MATLAB and SIMULINK with a step size of 0.5 sec and the results of the process of UCG with SSTA are compared with those of the first order sliding mode (FOSM) control.

The UCG reactor is provided with a constant flow rate of oxidants ($u = 2 \times 10^{-4}$moles/cm^{2}sec), at the injection well, till 20,000 secs, in order to build sufficient calorific value and prevent the saturation of the actuators (compressors). The controller is brought into loop after 20,000 sec.

In Fig. 5(a) the calorific value of the product gas attained at the production well is shown. This Figure elaborate that both algorithms (SSTA and FOSM) maintain a calorific value of 113.831 KJ/mol. However, the zoomed versions of the calorific values attained by SSTA and FOSM, shown in Figures 5(b) and 5(c) respectively, dic-

Table 1. Description of states and parameters.

Name	Description	Unit
x_1	Coal Density	g/cm^3
x_2	Char Density	g/cm^3
x_3	Solid Temperature	K
x_4	Concentration of CO	Moles/cm^3
x_5	Concentration of CO_2	Moles/cm^3
x_6	Concentration of H_2	Moles/cm^3
x_7	Concentration of CH_4	Moles/cm^3
x_8	Concentration of Tar	Moles/cm^3
x_9	Concentration of H_2O	Moles/cm^3
x_{10}	Concentration of O_2	Moles/cm^3
x_{11}	Concentration of N_2	Moles/cm^3
M_c	Molecular Weight of c Specie where c Represents Coal or Char	g/mole
$a_{i,j}$	Stoichiometric Coefficient of i^{th} Specie in j^{th} Chemical Reaction	-
r_j	Reaction Rates ($j = 1,2,3$) $r_1 = 5\frac{\rho_{coal}}{M_{coal}}\exp\left(\frac{-6039}{T_s}\right)$ $r_2 = \frac{1}{\frac{1}{r_{c_2}}+\frac{1}{r_{m_2}}}$ $r_3 = \frac{1}{\frac{1}{r_{c_3}}+\frac{1}{k_y m_{H_2O}}}$	-
C_s	Total Solid Phase Heat Capacity	Cal/K.cm^3
ht	Convective Heat Transfer Coefficient	Cal/sec.K.cm^3
T	Ignition Temperature	K
H_s	Solid Phase Heat Source	Cal/Sec.cm^3
L	Length of the Reactor	cm
mf_i	Mole Fraction of i^{th} Gas $\left(mf_i = \frac{x_i}{\sum_{i=4}^{11} x_i}\right)$	-
H_i	Heat of Combustion of i^{th} Gas	KJ/Mole
u	Flow Rate of Inlet Gases	Moles/cm^{2}Sec
h	Calorific Value of the Product Gas	KJ/Mole

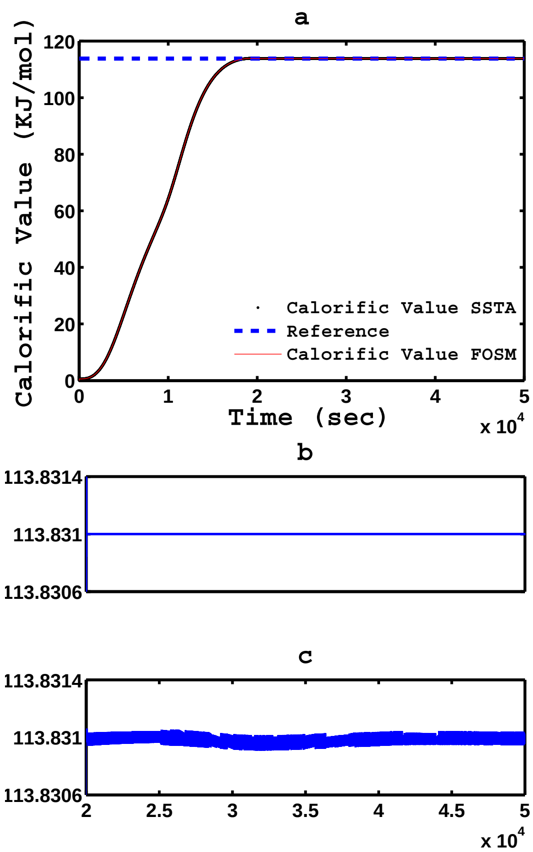

Fig. 5. (a) Reference tracking of the system for smooth STA and FOSM controller. (b) Zoomed version of calorific value attained by SSTA. (c) Zoomed version of calorific value attained by FOSM controller.

tate the robustness of SSTA. In Fig. 5(c) it can be seen that the tracking performance of FOSM is disturbed by the water influx (Fig. 3).

Figs. 6(a) and 6(b) respectively shows the control efforts (profile of flow rate of oxidants at the injection well) generated by FOSM control and SSTA. These figures clearly separate the performance of the two control strategies. SSTA proves to be a better one for following reasons. Control effort for SSTA is chattering free (**Smooth**) as compared to the high frequency oscillations in case of FOSM control. Moreover, the control effort for FOSM violates the constraint set by (29) when perturbation ζ (Fig. (3) reaches its peak value at approximately $33,000$ secs, whereas, the control effort produced by SSTA successfully copes the water influx, without violating constraint (compressor limitation).

Figs. 7(a) and 7(b) show the sliding surface reached by FOSM and SSTA respectively. In case of FOSM controller the effect of perturbation is reflected in sliding surface due to the saturation of control effort (Fig. 6(a)), where as the sliding surface for SSTA is unaffected by the

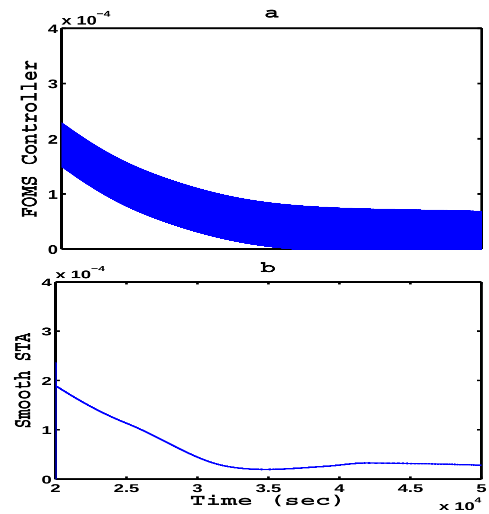

Fig. 6. (a) Molar Flow Rate ($moles/cm^2 Sec$) (control input) produced by FOSM (b) Molar Flow Rate produced by Smooth STA.

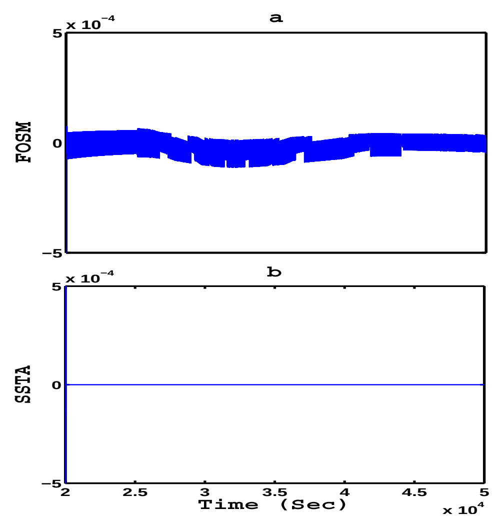

Fig. 7. a) Sliding surface (error of output and reference) system for FOSM. (b) Sliding surface for smooth STA.

perturbation and approximately stays at 0 KJ/mol (with sliding accuracy of 10^{-9} [6]), for whole simulation time.

5. CONCLUSION

In this paper the analytic expressions are proposed for gains and convergence time of SSTA for nominal as well as the perturbed system. These objectives are achieved via the introduction of a family of quadratic strict Lyapunov functions. The analytic expressions of the convergence time reveals the importance of the smoothing parameter ρ. The SSTA converges to the origin in finite time, when it is under the effect some bounded vanishing perturbations and ensures uniform ultimate boundedness when the perturbations are non-vanishing.

This analysis can be used for many purposes, it gives the freedom to design the gains of the controller for ensuring a desired performance. The estimate of the convergence time can be subjected to some optimization framework to further improve the performance. Also, this analysis can be useful for structural improvement of SSTA.

The SSTA with the proposed analytic expressions, is applied to the process of UCG. This is a highly nonlinear process with numerous model uncertainties. The simulation results reflect the effectiveness of the algorithm in terms of robustness and performance.

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